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SOLUTIONS

ALGEBRAIC PROOF OF ARITHMETIC RESULTS

SAMPLE QUESTIONS 1

(a) Prove algebraically that the sum of two odd numbers is always even.

Let the first odd number be $2n + 1$ and the other odd number be $2m + 1$

Adding gives:

$$\begin{aligned}(2n+1)+(2m+1) &= 2n+2m+2 \\ &= 2(n+m+1)\end{aligned}$$

This shows a factor of 2 so the result is always even.

(b) Prove algebraically that the product of two odd numbers is always odd.

Let the first odd number be $2n + 1$ and the other odd number be $2m + 1$

Multiplying gives:

$$\begin{aligned}(2n+1)(2m+1) &= 4mn+2n+2m+1 \\ &= 2(2mn+n+m)+1\end{aligned}$$

The expression can be written as one more than a multiple of 2, so it is always odd

(c) Prove algebraically that the difference between two odd numbers is always even.

Let the first odd number be $2n + 1$ and the other odd number be $2m + 1$

Subtracting gives:

$$\begin{aligned}(2n+1)-(2m+1) &= 2n+1-2m-1 \\ &= 2n-2m \\ &= 2(n-m)\end{aligned}$$

This shows a factor of 2 so the result is always even.

(d) Prove algebraically that the difference between an odd number and an even number is always odd.

Let the odd number be $2n + 1$ and the even number be $2m$

Subtracting gives:

$$\begin{aligned}(2n+1)-2m &= 2n-2m+1 \\ &= 2(n-m)+1\end{aligned}$$

The expression can be written as one more than a multiple of 2, so it is always odd

(e) Prove algebraically that the square of an even number is always even.

Let the even number be $2n$

Squaring gives:

$$(2n)^2 = 4n^2$$
$$= 2(2n^2)$$

This shows a factor of 2 so the result is always even.

(f) Prove algebraically that $7(n + 2) - 3(n + 4)$ is always even.

$$7(n+2)-3(n+4) = 7n+14-3n-12$$
$$= 4n+2$$
$$= 2(2n+1)$$

This shows a factor of 2 so the result is always even.

(g) Prove algebraically that $n^2 - 2 - (n - 2)^2$ is always an even number.

$$n^2-2-(n-2)^2 = n^2-2-\left[n^2-4n+4\right]$$
$$= n^2-2-n^2+4n-4$$
$$= 4n-6$$
$$= 2(2n-3)$$

This shows a factor of 2 so the result is always even.

(h) Prove algebraically that $(n + 1)^2 - (n - 1)^2 + 1$ is odd for all positive integer values of n.

$$(n+1)^2-(n-1)^2+1 = \left[n^2+2n+1\right]-\left[n^2-2n+1\right]+1$$
$$= n^2+2n+1-n^2+2n-1+1$$
$$= 4n+1$$

This is one more than a multiple of 2 so the result is an odd number.

(i) Prove algebraically that $(n + 3)(2n + 1) + (n - 2)(2n + 1)$ is **not** an even number.

$$(n+3)(2n+1)+(n-2)(2n+1) = \left[2n^2+n+6n+3\right]+\left[2n^2+n-4n-2\right]$$
$$= 2n^2+7n+3+2n^2-3n-2$$
$$= 4n^2+4n+1$$
$$= 2(2n^2+2n)+1$$

This is one more than a multiple of 2 so the result is odd, not even.

SAMPLE QUESTIONS 2

(a) Prove algebraically that the sum of any two consecutive odd numbers is always even.

Let the first odd number be $2n + 1$.

This means that the next odd number will be $2n + 3$ and their sum will be

$$\begin{aligned}(2n+1)+(2n+3) &= 4n+4 \\ &= 2(2n+2)\end{aligned}$$

This shows a factor of 2 so it is even.

(b) Prove algebraically that the difference between any two consecutive odd numbers is always two.

Let the first odd number be $2n + 1$.

This means that the next odd number will be $2n + 3$ and their difference will be

$$\begin{aligned}(2n+3)-(2n+1) &= 2n+3-2n-1 \\ &= 2\end{aligned}$$

(c) Prove algebraically that the product of any two consecutive even numbers is always even.

Let the first even number be $2n$.

This means that the next even number will be $2n + 2$ and their product will be:

$$\begin{aligned}2n(2n+2) &= 4n^2+4n \\ &= 2(2n^2+2n)\end{aligned}$$

This shows a factor of 2 so it is even

(d) Prove algebraically that the mean of three consecutive integers is always the middle number

Let the first integer be n.

This means that the next two integers will be $n + 1$ and $n + 2$

Their mean will be:

$$\begin{aligned}\frac{n+(n+1)+(n+2)}{3} &= \frac{3n+3}{3} \\ &= \frac{3(n+1)}{3} \\ &= n+1\end{aligned}$$

Which is the middle number

(e) Prove algebraically that the sum of any three consecutive even integers is always even.

Let the first even integer be $2n$.

This means that the next two even integers will be $2n + 2$ and $2n + 4$ and their sum will be:

$$\begin{aligned} 2n+(2n+2)+(2n+4) &= 6n+6 \\ &= 2(3n+3) \end{aligned}$$

This shows a factor of 2 so it is even

(f) Prove algebraically that the difference between the squares of any two consecutive integers is always an odd number.

Let the first integer be n.

This means that the next integer will be $n + 1$ and the difference between the squares is

$$\begin{aligned} (n+1)^2 - n^2 &= \left[n^2 + 2n + 1\right] - n^2 \\ &= 2n+1 \end{aligned}$$

This is one more than a multiple of two so it is an odd number.

SAMPLE QUESTIONS 3

(a) Prove algebraically that $5(n + 7) + 3(n + 3)$ is always a multiple of 4

$$\begin{aligned} 5(n+7)+3(n+3) &= 5n+35+3n+9 \\ &= 8n+44 \\ &= 4(2n+11) \end{aligned}$$

This shows a factor of 4 so the result is a multiple of 4.

(b) Prove algebraically that $8(n + 7) - 5(n + 4)$ is always a multiple of 3

$$\begin{aligned} 8(n+7)-5(n+4) &= 8n+56-5n-20 \\ &= 3n+36 \\ &= 3(n+12) \end{aligned}$$

This shows a factor of 3 so the result is a multiple of 3.

(c) Prove algebraically that $7(n + 8) + 5(n - 4)$ is always a multiple of 12

$$\begin{aligned} 7(n+8)+5(n-4) &= 7n+56+5n-20 \\ &= 12n+36 \\ &= 12(n+3) \end{aligned}$$

This shows a factor of 12 so the result is a multiple of 12.

(d) Prove algebraically that $(m + 2)^2 - m^2 - 12$ is always a multiple of 4

$$\begin{aligned} (m+2)^2-m^2-12 &= \left[m^2+4m+4\right]-m^2-12 \\ &= 4m-8 \\ &= 4(m-2) \end{aligned}$$

This shows a factor of 4 so the result is a multiple of 4.

(e) Prove algebraically that $(n + 6)^2 - (n + 2)^2$ is always a multiple of 8, for all positive integer values of n.

$$\begin{aligned} (n+6)^2-(n+2)^2 &= \left[n^2+12n+36\right]-\left[n^2+4n+4\right] \\ &= n^2+12n+36-n^2-4n-4 \\ &= 8n+32 \\ &= 8(n+4) \end{aligned}$$

This shows a factor of 8 so the result is a multiple of 8.

(f) Prove algebraically that $(5n - 3)^2 - 3(3 - 10n)$ is always a multiple of 5

$$\begin{aligned} (5n-3)^2-3(3-10n) &= \left[25n^2-30n+9\right]-9+30n \\ &= 25n^2 \\ &= 5\times 5n^2 \end{aligned}$$

This shows a factor of 5 so the result is a multiple of 5.

(g) Prove algebraically that $(4n + 2)^2 - 12(n + 1)$ is always a multiple of 4

$$(4n+2)^2 - 12(n+1) = \left[16n^2 + 16n + 4\right] - 12n - 12$$
$$= 16n^2 + 4n - 8$$
$$= 4(4n^2 + n - 2)$$

This shows a factor of 4 so the result is a multiple of 4.

(h) Prove algebraically that $(4n + 1)^2 - (4n - 1)^2$ is always a multiple of 8, for all positive integer values of n.

$$(4n+1)^2 - (4n-1)^2 = \left[16n^2 + 8n + 1\right] - \left[16n^2 - 8n + 1\right]$$
$$= 16n^2 + 8n + 1 - 16n^2 + 8n - 1$$
$$= 16n$$
$$= 8(2n)$$

This shows a factor of 8 so the result is a multiple of 8.

(i) Show that when x is a whole number $7(2x + 1) + 6(x + 3)$ is always a multiple of 5

$$7(2x+1) + 6(x+3) = 14x + 7 + 6x + 18$$
$$= 20x + 25$$
$$= 5(4x + 5)$$

This shows a factor of 5 so the result is a multiple of 5.

(j) Prove algebraically that the sum of three <u>consecutive odd numbers</u> is always a multiple of 3

Let the first odd number be $2n + 1$.

This means that the next two odd numbers will be $2n + 3$ and $2n + 5$ and their sum will be:

$$(2n+1) + (2n+3) + (2n+5) = 6n + 9$$
$$= 3(2n + 3)$$

This shows a factor of 3 so the result is a multiple of 3.

(k) Prove that $(2n + 1)^2 - (2n - 1)^2 - 2$ is **not** a multiple of 4 for all positive integer values of n.

$$(2n+1)^2 - (2n-1)^2 - 2 = \left[4n^2 + 4n + 1\right] - \left[4n^2 - 4n + 1\right] - 2$$
$$= 4n^2 + 4n + 1 - 4n^2 + 4n - 1 - 2$$
$$= 8n - 2$$

Thus the expression does not have a factor of 4, so the result is not a multiple of 4.

MIXED QUESTIONS

(a) Prove algebraically that the sum of the squares of **any** two **odd** positive integers is always even.

If the first odd integer is $2n + 1$ and the second odd integer is $2m + 1$, then the sum of their squares will be:

$$\begin{aligned}(2n+1)^2+(2m+1)^2 &= \left[4n^2+4n+1\right]+\left[4m^2+4m+1\right]\\ &= 4n^2+4m^2+4n+4m+2\\ &= 2(2n^2+2m^2+2n+2m+1)\end{aligned}$$

This shows a factor of 2 so the result is always even.

(b) Prove that the sum of the squares of two consecutive **odd** numbers is always 2 more than a multiple of 8.

If the first odd integer is $2n + 1$, then the second odd integer will be $2n + 3$, and the sum of their squares will be:

$$\begin{aligned}(2n+1)^2+(2n+3)^2 &= \left[4n^2+4n+1\right]+\left[4n^2+12n+9\right]\\ &= 8n^2+16n+10\\ &= 8n^2+16n+8+2\\ &= 8(n^2+2n+1)+2\end{aligned}$$

Which is two more than a multiple of 8.

(c) Prove algebraically that the sum of any two consecutive **odd** numbers is always a multiple of 4.

If the first odd integer is $2n + 1$, then the second odd integer will be $2n + 3$, and their sum will be:

$$\begin{aligned}(2n+1)+(2n+3) &= 4n+4\\ &= 4(n+1)\end{aligned}$$

This shows a factor of 4 so the result is a multiple of 4.

(d) Prove that the sum of the squares of any three consecutive odd numbers is always 11 more than a multiple of 12.

If the first odd integer is $2n + 1$, then the second odd integer will be $2n + 3$, and the third will be $2n + 5$. The sum of their squares will be:

$$\begin{aligned}(2n+1)^2+(2n+3)^2+(2n+5)^2 &= \left[4n^2+4n+1\right]+\left[4n^2+12n+9\right]+\left[4n^2+20n+25\right]\\ &= 12n^2+36n+35\\ &= 12n^2+36n+24+11\\ &= 12(n^2+3n+2)+11\end{aligned}$$

Which is 11 more than a multiple of 12.

(e) Prove algebraically that $n^2 - 2 - (n - 2)^2$ is always an even number.

$$\begin{aligned} n^2-2-(n-2)^2 &= n^2-2-\left[n^2-4n+4\right] \\ &= n^2-2-n^2+4n-4 \\ &= 4n-6 \\ &= 2(2n-3) \end{aligned}$$

This shows a factor of 2 so the result is an even number.

(f) Prove that $(3n + 1)^2 – (3n – 1)^2$ is always a multiple of 12, for all positive integer values of n.

$$\begin{aligned} (3n+1)^2-(3n-1)^2 &= \left[9n^2+6n+1\right]-\left[9n^2-6n+1\right] \\ &= 9n^2+6n+1-9n^2+6n-1 \\ &= 12n \end{aligned}$$

This shows a factor of 12 so the result is a multiple of 12.

(g) Prove that $(8n + 2)^2 – (8n – 3)^2$ is always a multiple of 5

$$\begin{aligned} (8n+2)^2-(8n-3)^2 &= \left[64n^2+32n+4\right]-\left[64n^2-48n+9\right] \\ &= 64n^2+32n+4-64n^2+48n-9 \\ &= 80n-5 \\ &= 5(16n-1) \end{aligned}$$

This shows is a factor of 5 so the result is a multiple of 5.

(h) Prove algebraically that the sum of four consecutive integers is **not** divisible by 4.

If the first integer is n, then the next three integers will be $n + 1$, $n + 2$ and $n + 3$

Their sum will be:

$$n+(n+1)+(n+2)+(n+3)=4n+6$$

$4n + 6$ does not have *a factor of 4 so the result is not a multiple of 4 and is therefore not divisible by 4.*

(i) Prove that $(n – 1)^2 + n^2 + (n + 1)^2 = 3n^2 + 2$

$$\begin{aligned} (n-1)^2+n^2+(n+1)^2 &= \left[n^2-2n+1\right]+n^2+\left[n^2+2n+1\right] \\ &= 3n^2+2 \end{aligned}$$

EXTENSION 1

(a) Prove that an odd number cubed is also odd.

Let the odd number be $2n + 1$ so cubing it will give:

$$\begin{aligned}(2n+1)^3 &= (2n+1)(2n+1)(2n+1)\\ &= (2n+1)\left[4n^2+4n+1\right]\\ &= 8n^3+8n^2+2n+4n^2+4n+1\\ &= 8n^3+12n^2+6n+1\\ &= 2(4n^3+6n^2+3n)+1\end{aligned}$$

This is one more than a multiple of 2 so it is odd.

(b) Prove that the sum of two consecutive multiples of 5 is always an odd number.

Let the first multiple of five be $5n$.

This means that the next multiple of five will be $5n + 5$ and their sum will be:

$$\begin{aligned}5n+(5n+5) &= 10n+5\\ &= 10n+4+1\\ &= 2(5n+2)+1\end{aligned}$$

This is one more than a multiple of 2 so it is odd.

(c) Given that $2(x - n) = x + 5$ where n is an integer, prove that x must be an odd number.

$$2(x-n) = x+5$$
$$\Rightarrow \quad 2x-2n = x+5$$
$$\Rightarrow \quad x = 2n+5$$
$$\Rightarrow \quad x = 2n+4+1$$
$$\Rightarrow \quad x = 2(n+2)+1$$

This shows that x is one more than a multiple of 2 so x is odd.

(d) Given that $4(x + n) = 3x + 10$ where n is an integer, prove that x must be an even number.

$$4(x+n) = 3x+10$$
$$\Rightarrow \quad 4x+4n = 3x+10$$
$$\Rightarrow \quad x = 4n+10$$
$$\Rightarrow \quad x = 2(2n+5)$$

This shows that x has a factor of 2 so x is even.

(e) Prove that if the difference of two integers is 4, then the difference of their squares is a multiple of 8

Let the integers be n and m.

If their difference if 4, then:

$$
\begin{aligned}
n - m &= 4 \\
\Rightarrow \quad n &= m + 4 \\
\Rightarrow \quad n^2 - m^2 &= (m+4)^2 - m^2 \\
&= m^2 + 8m + 16 - m^2 \\
&= 8m + 16 \\
&= 8(m+2)
\end{aligned}
$$

This shows a factor of 8 so the answer is always a multiple of 8.

(f) Prove that for any three consecutive numbers, the difference between the squares of the first and last numbers is 4 times the middle number.

Let the first integer be n.

This means that the middle integer will be $n + 1$ and the last integer will be $n + 2$

So the difference between the squares of the first and last numbers will be:

$$
\begin{aligned}
(n+2)^2 - n^2 &= n^2 + 4n + 4 - n^2 \\
&= 4n + 4 \\
&= 4(n+1)
\end{aligned}
$$

Which is 4 times the middle number.

(g) If n is a positive integer greater than 1, prove that $n^3 - n$ is a multiple of 6 for all possible values of n.

$$
\begin{aligned}
n^3 - n &= n(n^2 - 1) \\
&= n(n+1)(n-1)
\end{aligned}
$$

This is the product of three consecutive integers.

With three consecutive integers, at least one of them must be even and so the result must be a multiple of 2.

With three consecutive integers, exactly one will be a multiple of three so the result must also be a multiple of 3.

If the result is both a multiple of 2 and a multiple of 3, then it must be a multiple of 6.

(h) Prove that the difference between the squares of any two consecutive integers is equal to the sum of the two integers.

Let the first integer be n.

This means the next integer will be $n + 1$ and the difference between their squares will be:

$$\begin{aligned}(n+1)^2 - n^2 &= n^2 + 2n + 1 - n^2 \\ &= 2n + 1 \\ &= n + n + 1 \\ &= n + (n+1)\end{aligned}$$

Which is the sum of the two integers.

(i) Prove that the sum of the squares of two consecutive odd numbers is always 2 more than a multiple of 8

Let the first odd number be $2n + 1$

This means that the next odd number will be $2n + 3$ and the sum of their squares will be:

$$\begin{aligned}(2n+1)^2 + (2n+3)^2 &= \left[4n^2 + 4n + 1\right] + \left[4n^2 + 12n + 9\right] \\ &= 8n^2 + 16n + 10 \\ &= 8n^2 + 16n + 8 + 2 \\ &= 8(n^2 + 2n + 1) + 2\end{aligned}$$

Which is two more than a multiple of 8.

(j) The product of two consecutive positive integers is added to the larger of the two integers. Prove that the result is always a square number.

Let the first integer be n.

This means that the next (larger) integer will be $n + 1$ and so:

$$\begin{aligned}n(n+1) + (n+1) &= n^2 + n + n + 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2\end{aligned}$$

Which is the square of the larger number.

EXTENSION 2

(a) If a, b and c are three consecutive integers, prove that $c^2 - a^2 = 4b$

If the smallest integer is a then $b = a + 1$ and $c = a + 2$

Therefore:

$$\begin{aligned} c^2 - a^2 &= (a+2)^2 - a^2 \\ &= \left[a^2 + 4a + 4\right] - a^2 \\ &= 4a + 4 \\ &= 4(a+1) \\ &= 4b \end{aligned}$$

(b) Given that n is an integer, prove algebraically that the sum of $(n + 2)(n + 1)$ and $n + 2$ is always a square number.

$$\begin{aligned} (n+1)(n+2) + (n+2) &= \left[n^2 + 3n + 2\right] + (n+2) \\ &= n^2 + 4n + 4 \\ &= (n+2)^2 \end{aligned}$$

Which is a square number.

(c) Prove that for any two numbers the product of their difference and their sum is equal to the difference of their squares.

Let the first number be n and the second number be m.

This means that the product of their difference and sum is:

$$\begin{aligned} (m-n)(m+n) &= m^2 + mn - mn - n^2 \\ &= m^2 - n^2 \end{aligned}$$

Which is the difference of their squares.

(d) If a, b, c and d are four consecutive integers, show that the product of the first and last is two less than the product of the second and third.

If the smallest integer is a then $b = a + 1$, $c = a + 2$ and $d = a + 3$

Therefore the product of the first and last is:

$$\begin{aligned} ad &= a(a+3) \\ &= a^2 + 3a \\ &= a^2 + 3a + 2 - 2 \\ &= (a+1)(a+2) - 2 \\ &= bc - 2 \end{aligned}$$

Which is two less than the product of the second and third.

(e) Given that n is an integer, prove that $(n – 2)(n + 3) + (6 – n)$ is a square number.

$$(n-2)(n+3)+(6-n)=\left[n^2+3n-2n-6\right]+6-n$$
$$=n^2$$

So the result is a square number.

(f) Given that triangle numbers can be represented by $T_n=\frac{n(n+1)}{2}$.

Prove that eight times **any** triangle number is one less than a square number.

$$8\times\frac{n(n+1)}{2}=4n(n+1)$$
$$=4n^2+4n$$
$$=\left[4n^2+4n+1\right]-1$$
$$=(2n+1)(2n+1)-1$$
$$=(2n+1)^2-1$$

Which is one less than a square number.

(g) n is an integer.

Prove algebraically that the sum of $\frac{1}{2}n(n+1)$ and $\frac{1}{2}(n+1)(n+2)$ is always a square number.

$$\frac{1}{2}n(n+1)+\frac{1}{2}(n+1)(n+2)=\frac{1}{2}\left[n^2+n\right]+\frac{1}{2}\left[n^2+3n+2\right]$$
$$=\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{2}n^2+\frac{3}{2}n+1$$
$$=n^2+2n+1$$
$$=(n+1)(n+1)$$
$$=(n+1)^2$$

Which is a square number.

(h) x is a positive whole number.
Explain why the expression $2x^2 + 5x + 2$ can never have a value that is a prime number.

$$2x^2+5x+2=(2x+1)(x+2)$$

Thus, the expression can be written as the product of two integers both of which are greater than one. This means that it will only ever evaluate to a composite number, never a prime.

(i) $2^{61} – 1$ is a prime number. Explain why $2^{61} + 1$ must be a multiple of 3.

$2^{61}-1$, 2^{61} *and* $2^{61}+1$ *are three consecutive integers so one of them must be a multiple of* 3

$2^{61}-1$ *is not the multiple of* 3 *because it is prime.*

2^{61} *is not the multiple of* 3 *because it only has factors which are powers of* 2

This means that $2^{61}-1$ *must be the multiple of* 3