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WORKBOOK

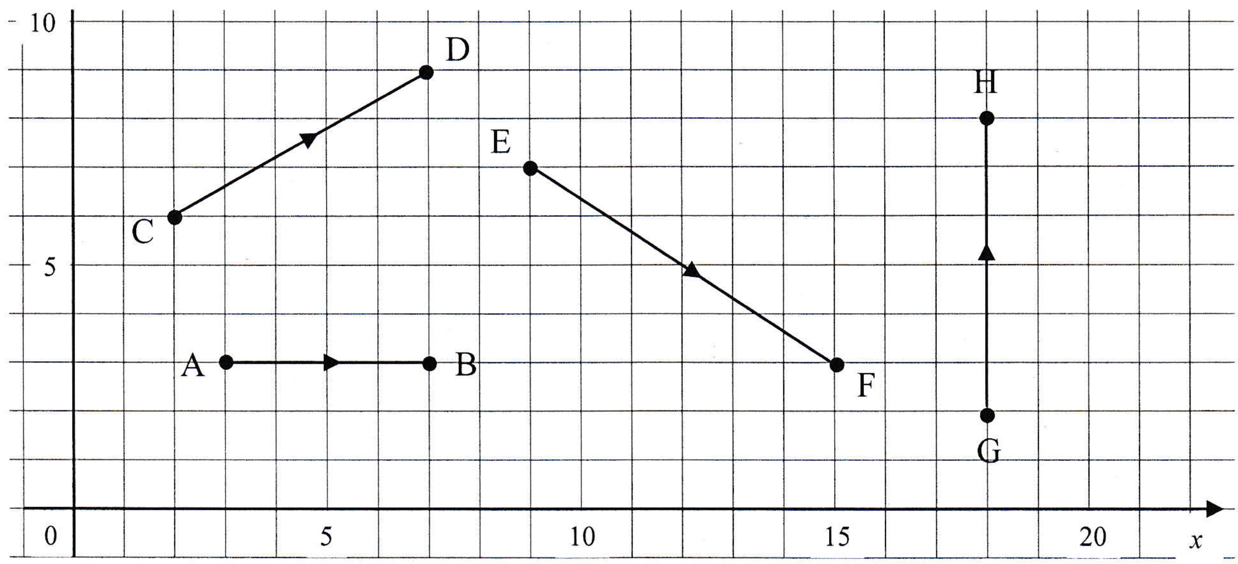
INTRODUCTION TO VECTORS

COLUMN VECTORS

A vector is a quantity that has both **magnitude** (size) **and direction**.

Force & *velocity* are commonly used vectors in physics, but we focus on 'displacement' vectors in GCSE maths – these give the magnitude and direction of **a movement from one point to another**.

The following diagram shows four (displacement) vectors on a grid:



Notation:

The notation AB represents the line that is drawn between A and B.
The notation $\overrightarrow{AB}$ represents the vector displacement from A to B.

Vector displacements can be described by the change in their horizontal and vertical coordinates – to distinguish a vector from actual coordinates, these horizontal and vertical changes are written in a column, one above the other (x above y).

Examples:

The vectors on the grid above can be described by the following column vectors:

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \qquad \overrightarrow{CD} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \qquad \overrightarrow{EF} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \qquad \overrightarrow{GH} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

Since vectors have direction, every vector has a 'reverse' vector which is in the opposite direction:

$$\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \qquad \overrightarrow{DC} = \begin{pmatrix} -5 \\ -3 \end{pmatrix} \qquad \overrightarrow{FE} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} \qquad \overrightarrow{HG} = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$$

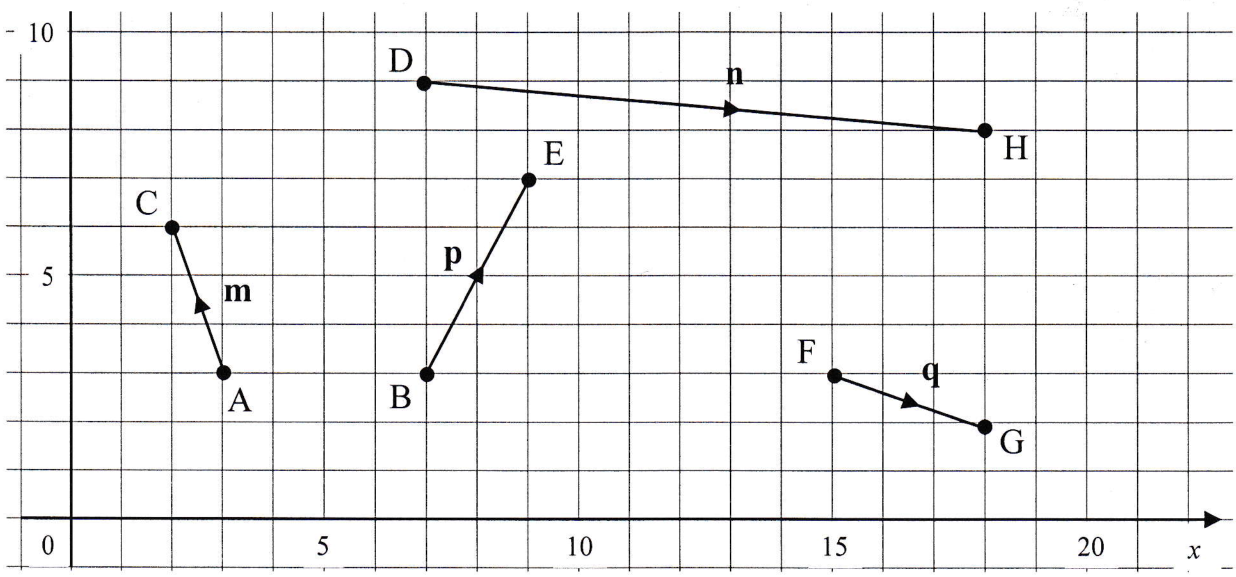
Notes:

The starting and finishing points of a vector are represented using uppercase letters, but vectors themselves are often represented using lowercase letters.

- Within text books (and other printed material) these lowercase letters are printed in **bold**;
- Within hand-written text, vectors are often indicated by using a squiggly underline – e.g. $\underset{\sim}{a}$

The following diagram shows some different vectors on a grid.

This time the vectors have been labelled using lowercase letters:



$\mathbf{m} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$ $\mathbf{p} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$-\mathbf{m} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $-\mathbf{n} = \begin{pmatrix} -11 \\ 1 \end{pmatrix}$

QUICK TASK:

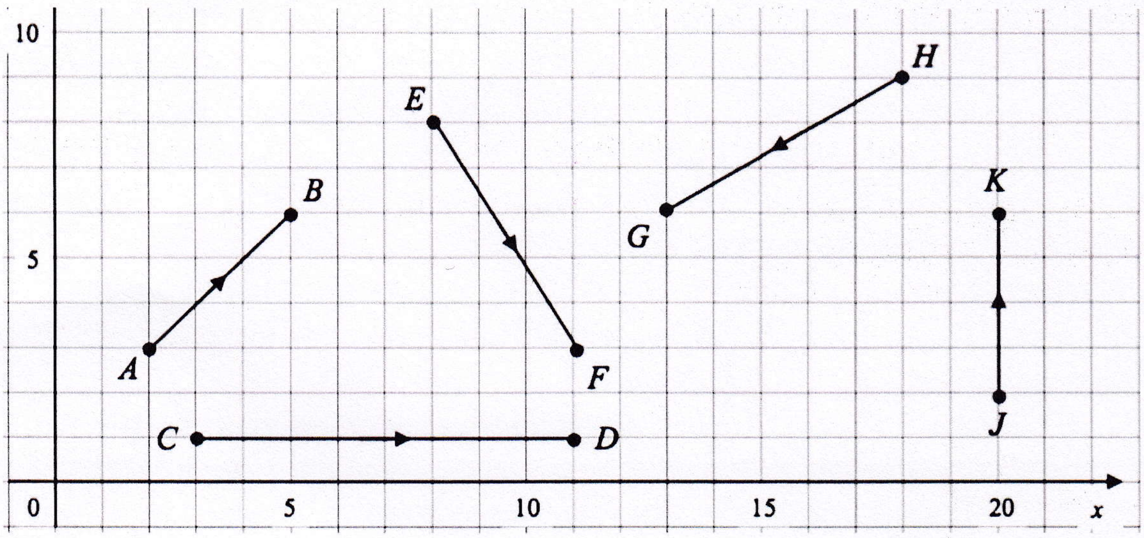
Write the vectors for…

(a) $-\mathbf{p}$ $= \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

(b) $-\mathbf{q}$ $= \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

PRACTICE QUESTIONS 1

(a) The following diagram shows some vectors on a grid:

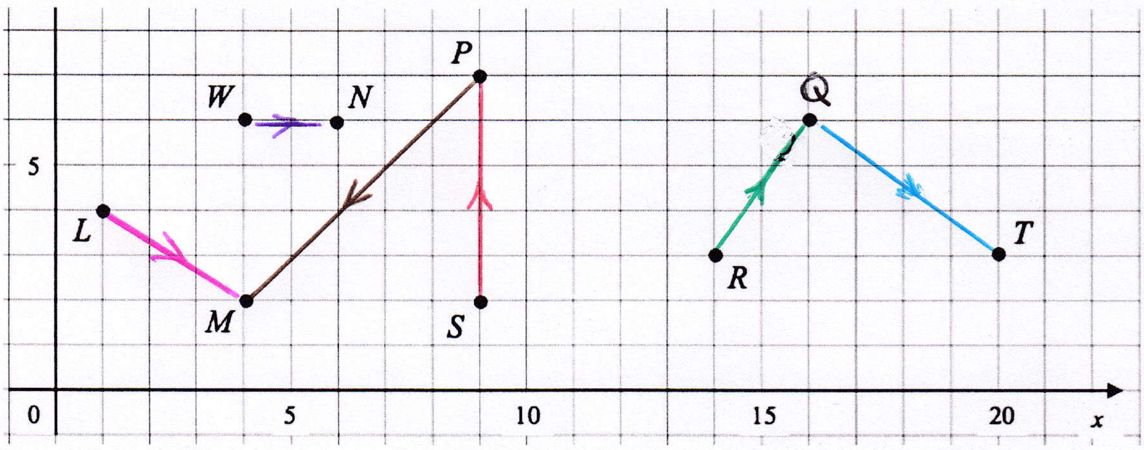


Write the column vectors for each of the following:

(i) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ (ii) $\overrightarrow{CD} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ (iii) $\overrightarrow{EF}$ $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

(iv) $\overrightarrow{HG}$ $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$ (v) $\overrightarrow{JK}$ $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (vi) $\overrightarrow{GH}$ $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

(b) The following diagram shows some points on a grid:



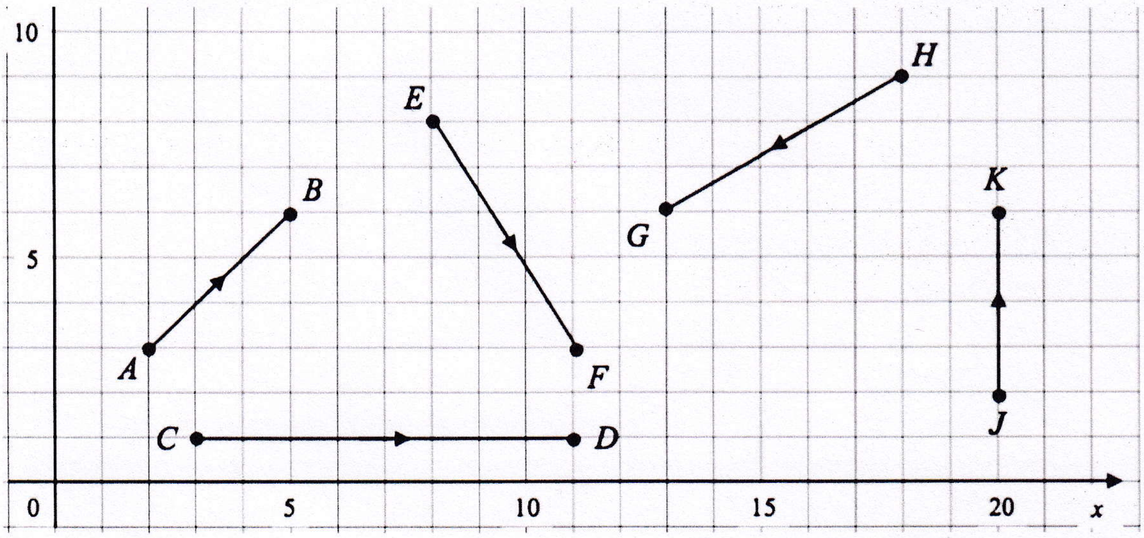
Draw each of the following vectors onto the grid and then write them as column vectors:

(i) $\overrightarrow{LM} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ (iv) $\overrightarrow{WN} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\overrightarrow{SP} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$

(ii) $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (v) $\overrightarrow{QT} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ $\overrightarrow{PM} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}$

PRACTICE QUESTIONS 1

(a) The following diagram shows some vectors on a grid:

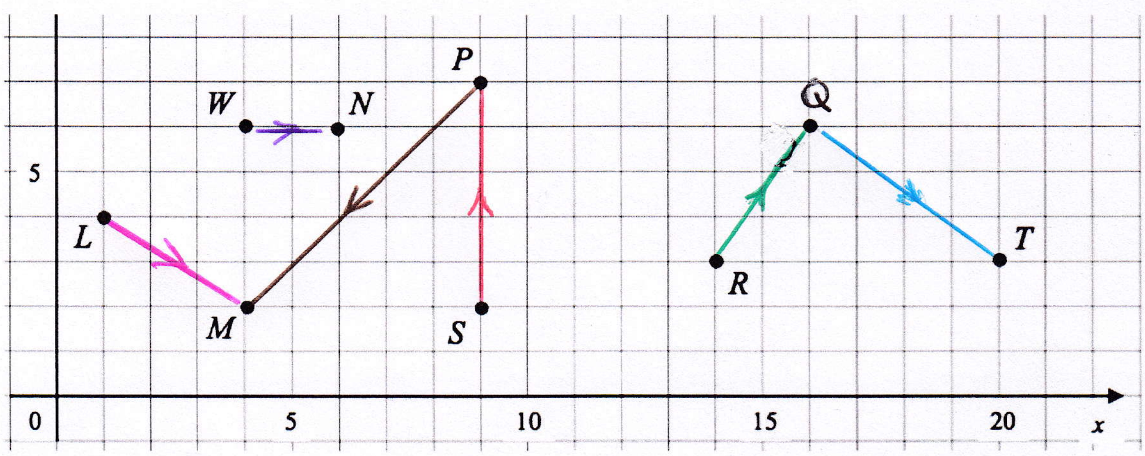


Write the column vectors for each of the following:

(i) $\overrightarrow{AB} = \begin{pmatrix}3\\3\end{pmatrix}$ (ii) $\overrightarrow{CD} = \begin{pmatrix}8\\0\end{pmatrix}$ (iii) $\overrightarrow{EF}$ $\begin{pmatrix}3\\-5\end{pmatrix}$

(iv) $\overrightarrow{HG}$ $\begin{pmatrix}-5\\-3\end{pmatrix}$ (v) $\overrightarrow{JK}$ $\begin{pmatrix}0\\4\end{pmatrix}$ (vi) $\overrightarrow{GH}$ $\begin{pmatrix}5\\3\end{pmatrix}$

(b) The following diagram shows some points on a grid:



Draw each of the following vectors onto the grid and then write them as column vectors:

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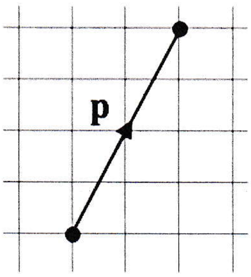
MAGNITUDE OF A VECTOR

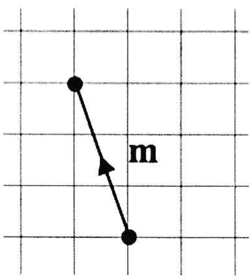
The magnitude of a vector is its length.

This can be worked out using Pythagoras.

Examples:



$\mathbf{p} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ Magnitude of **p** is $\sqrt{2^2 + 4^2} = 4.47$



$\mathbf{m} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ Magnitude of **m** is $\sqrt{(-1)^2 + 3^2} = 3.16$

In summary, the magnitude of the vector $\begin{pmatrix} \pm x \\ \pm y \end{pmatrix}$ is $\sqrt{x^2 + y^2}$

PRACTICE QUESTIONS 2

Calculate the magnitude of the following vectors.
Give your answers correct to 3 significant figures, when appropriate.

(i) $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \sqrt{3^2+2^2}$
$= 3.61$

(ii) $\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \sqrt{5^2+2^2}$
$= 5.39$

(iii) $\begin{pmatrix} 4 \\ 0 \end{pmatrix} = 4$

HORIZONTAL LINE

(iv) $\begin{pmatrix} -3 \\ -5 \end{pmatrix} = \sqrt{3^2+5^2}$
$= 5.83$

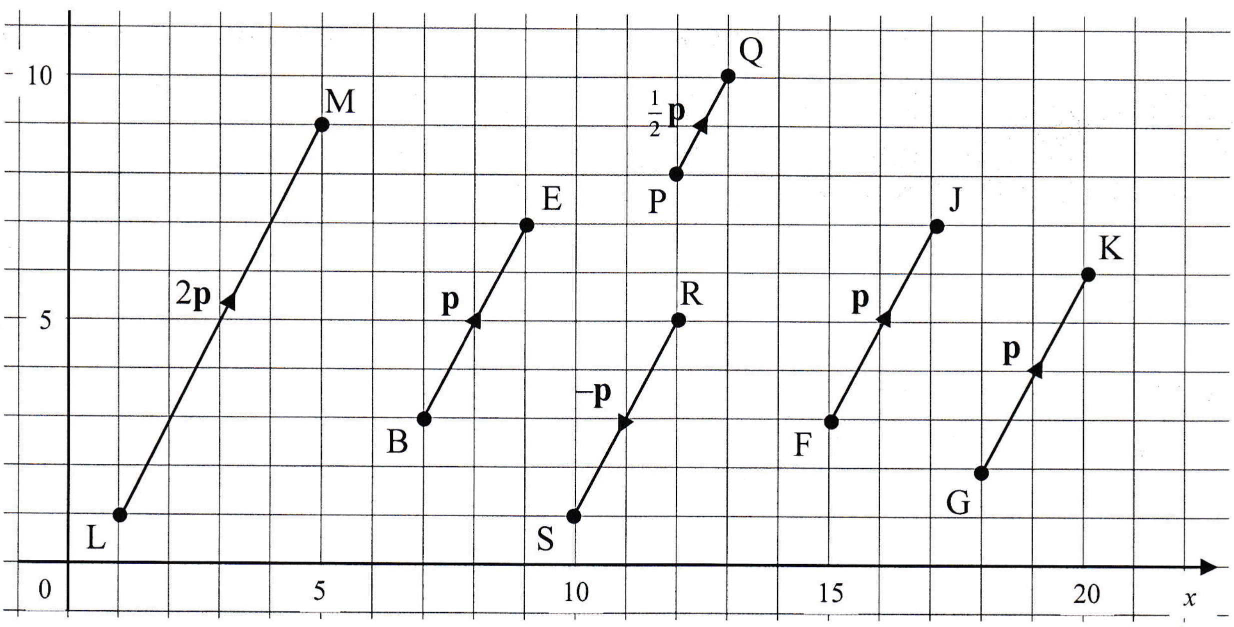
(v) $\begin{pmatrix} 0 \\ 7 \end{pmatrix} = 7$

VERTICAL LINE

(vi) $\begin{pmatrix} -6 \\ 6 \end{pmatrix} = \sqrt{6^2+6^2}$
$= 8.49$

PARALLEL VECTORS

If two vectors are parallel then they either represent exactly the same displacement, or **one will be a multiple of the other**:



Note that $\overrightarrow{BE} = \overrightarrow{FJ} = \overrightarrow{GK} = \mathbf{p}$

Multiplying vectors:

The diagrams above show that:

$\overrightarrow{LM} = 2p$	$\overrightarrow{PQ} = \frac{1}{2}p$	$\overrightarrow{RS} = -p$
$= 2 \times \begin{pmatrix} 2 \\ 4 \end{pmatrix}$	$= \frac{1}{2} \times \begin{pmatrix} 2 \\ 4 \end{pmatrix}$	$= -\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
$= \begin{pmatrix} 4 \\ 8 \end{pmatrix}$	$= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$= \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

Note:

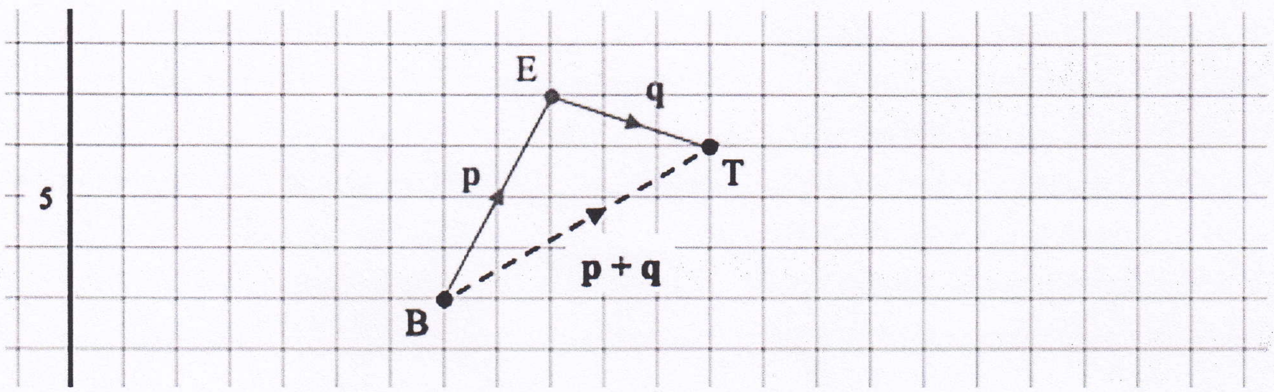
> If one vector is a multiple of another vector, then the two vectors must be parallel.

And:

> If one vector is a multiple of another vector and they **have a point in common**, then the two vectors must form a straight line.

ADDITION OF VECTORS

Standard addition:



$$p+q=\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$$

> The answer to a vector addition is called the **resultant**

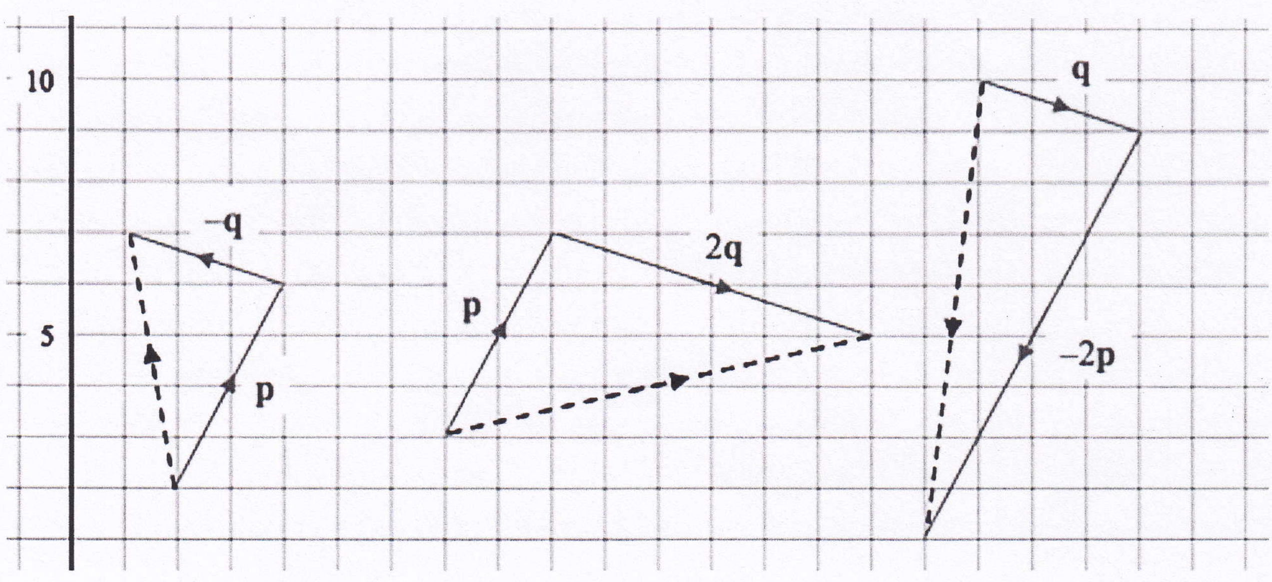
Other 'additions':

$$p-q=\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}-1\\5\end{pmatrix}$$

$$p+2q=\begin{pmatrix}2\\4\end{pmatrix}+2\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}6\\-2\end{pmatrix}=\begin{pmatrix}8\\2\end{pmatrix}$$

$$q-2p=\begin{pmatrix}3\\-1\end{pmatrix}-2\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix}-\begin{pmatrix}4\\8\end{pmatrix}=\begin{pmatrix}-1\\-9\end{pmatrix}$$

These additions are shown on the diagram below:



PRACTICE QUESTIONS 3

(a) If $a = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ $b = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ $c = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

Write as column vectors:

(i) $a + b$ $= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

(ii) $a + c$ $= \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(iii) $2a$ $= 2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$

(iv) $3c$ $= 3\begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \end{pmatrix}$

(v) $2a - 3b$ $= 2\begin{pmatrix} 1 \\ 4 \end{pmatrix} - 3\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix} - \begin{pmatrix} -15 \\ 9 \end{pmatrix} = \begin{pmatrix} 17 \\ -1 \end{pmatrix}$

(vi) $2b - c$ $= 2\begin{pmatrix} -5 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ 9 \end{pmatrix}$

(b) If $p = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ $q = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$ $r = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Find:

(i) the magnitude of p $= \sqrt{4^2 + 2^2}$

$= 4.47$

(ii) $2r - q$ as a column vector

$= 2\begin{pmatrix} -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$

(iii) the magnitude of $q + r$

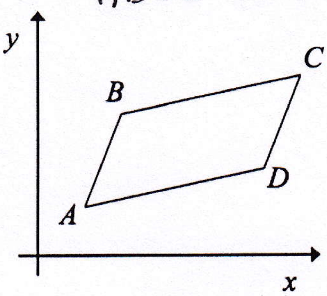
$q + r = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

$\therefore$ MAGNITUDE $= \sqrt{3^2 + 6^2}$

$= 6.71$

ABCD IS A PARALLELOGRAM

(c)



$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\overrightarrow{CB} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$

$\overrightarrow{DC} = \overrightarrow{AB}$

$\overrightarrow{DA} = \overrightarrow{CB}$

(i) Find $\overrightarrow{AC}$ as a column vector.

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

(ii) Find the magnitude of $\overrightarrow{BD}$.

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$= \begin{pmatrix} 7 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix} \Rightarrow$ MAGNITUDE $= \sqrt{6^2+1^2}$

$= 6.08$

(d) State what you can deduce from each of the following:

(i) $\overrightarrow{\mathrm{LM}} = b$ and $\overrightarrow{\mathrm{NP}} = 3b$

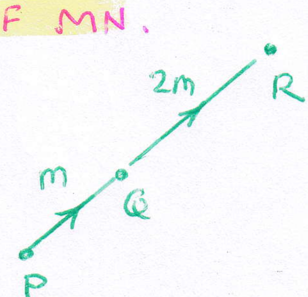
LM IS PARALLEL TO NP

NP IS THREE TIMES LARGER THAN LM

(ii) $\overrightarrow{\mathrm{MN}} = w$ and $\overrightarrow{\mathrm{ST}} = -2w$

ST IS PARALLEL TO MN, BUT IN THE OPPOSITE DIRECTION

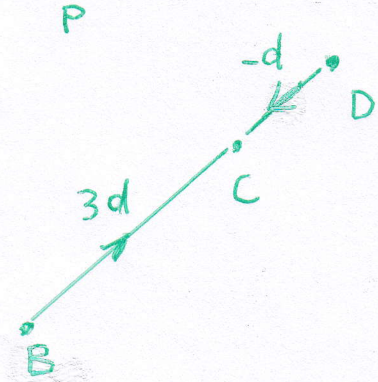
ST IS TWICE THE SIZE/LENGTH OF MN.



(iii) $\overrightarrow{\mathrm{PQ}} = m$ and $\overrightarrow{\mathrm{QR}} = 2m$

PQR IS A STRAIGHT LINE

QR IS TWICE AS LONG AS PQ



(iv) $\overrightarrow{\mathrm{BC}} = 3d$ and $\overrightarrow{\mathrm{DC}} = -d$

BCD IS A STRAIGHT LINE

BC IS THREE TIMES LONGER THAN DC

DC IS IN OPPOSITE DIRECTION TO BC

CORDINATES AND VECTORS

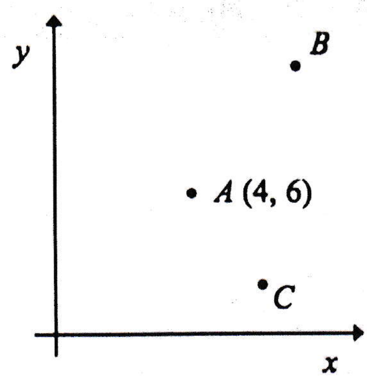
Note the difference between coordinates and (displacement) vectors:

coordinates give the **location** of a point on a grid

vectors give the **change in the location** of points.

Example 1:

Suppose that $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$



If the point A has coordinates (4, 6) then:

B will have coordinates $(4 + 3, 6 + 5) = (8, 11)$

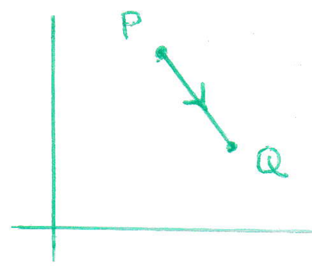
C will have coordinates $(4 + 2, 6 + -3) = (6, 3)$

Example 2:

If point P has coordinates (7, 13) and point Q has the coordinates (9, 6)

Then the vector $\overrightarrow{PQ} = \begin{pmatrix} 9-7 \\ 6-13 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -7 \end{pmatrix}$

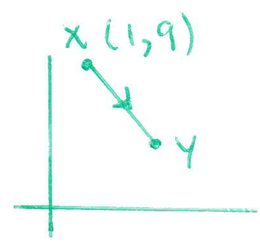


PRACTICE QUESTIONS 4

(a) X is the point (1, 9).

$\overrightarrow{XY} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $Y = (1+3, 9-5)$

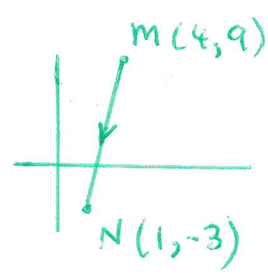
$= (4, 4)$



Find the coordinates of Y.

(b) If M is the point (4, 9) and N is the point (1, –3).

Find the vector $\overrightarrow{MN}$. $= (1-4, -3-9) = (-3, -12)$



(c) If the point $D = (1, 6)$ and $\overrightarrow{DE} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\overrightarrow{FE} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

Find the coordinates of F.

$\overrightarrow{DE} + \overrightarrow{EF} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ $\therefore F = (1+5, 6+-3)$

$= (6, 3)$

VECTOR DIAGRAMS

Most of the time at GCSE, we are not concerned with column vectors – we are concerned with finding resultants of vectors using vector diagrams.

In this type of question, you will usually be given a diagram showing interconnections between different points (usually in the shape of a triangle, trapezium, parallelogram etc.). You will also be given two vector displacements, which will be identified by letters.

Your task will be to work out other displacements in terms of the two vectors that are given.

This is best shown by studying the examples that follow.

Example 1

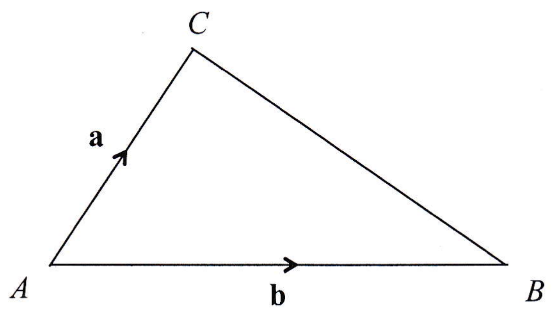


Diagram **NOT** accurately drawn

ABC is a triangle.

$\overrightarrow{AC} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$

Find, in terms of **a** and **b**

(i) $\overrightarrow{CB}$

(ii) $\overrightarrow{BC}$

Solution 1:

(i) $\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$

SAME POINT IS THE END OF ONE VECTOR AND START OF NEXT

$= -\mathbf{a} + \mathbf{b}$

(ii) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\mathbf{b} + \mathbf{a}$

Example 2

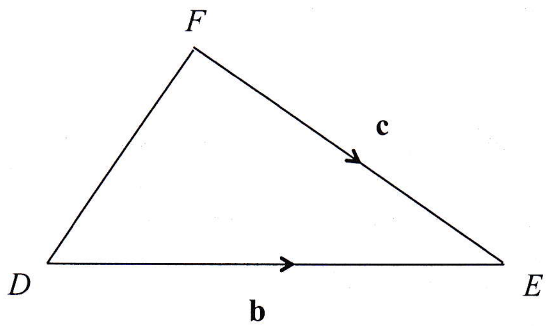


Diagram **NOT** accurately drawn

DEF is a triangle.

$\overrightarrow{DE} = \mathbf{b}$ and $\overrightarrow{FE} = \mathbf{c}$

Find, in terms of $\mathbf{b}$ and $\mathbf{c}$

(i) $\overrightarrow{DF}$

(ii) $\overrightarrow{FD}$

Solution 2:

(i) $\overrightarrow{DF} = \overrightarrow{DE} + \overrightarrow{EF}$

$= \mathbf{b} - \mathbf{c}$

(ii) $\overrightarrow{FD} = \overrightarrow{FE} + \overrightarrow{ED}$

$= \mathbf{c} - \mathbf{b}$

Example 3

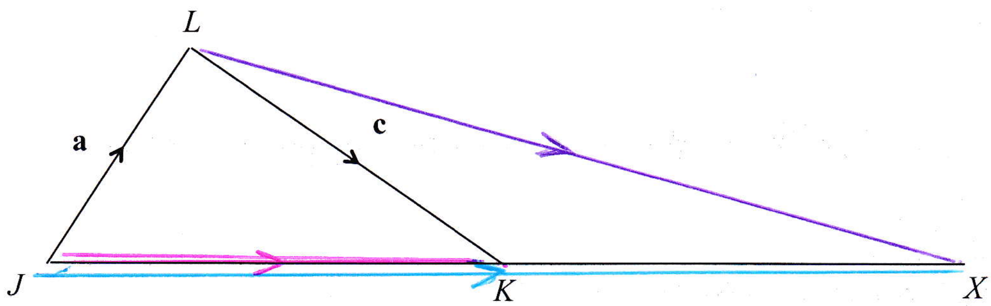


Diagram **NOT** accurately drawn

JKL is a triangle.

$\overrightarrow{JL} = \mathbf{a}$ and $\overrightarrow{LK} = \mathbf{c}$

$\boxed{JK = KX}$ IMPORTANT !

Find, in terms of **a** and **c**

(i) $\overrightarrow{JK}$

(iii) $\overrightarrow{JX}$

(iii) $\overrightarrow{LX}$

Solution 3:

(i)
$$\boxed{\overrightarrow{JK}} = \overrightarrow{JL} + \overrightarrow{LK}$$
$$= \mathbf{a} + \mathbf{c}$$

(ii)
$$\boxed{\overrightarrow{JX}} = 2 \times \overrightarrow{JK}$$
$$= 2(\mathbf{a} + \mathbf{c})$$
$$= 2\mathbf{a} + 2\mathbf{c}$$

(iii)
$$\boxed{\overrightarrow{LX}} = \overrightarrow{LJ} + \overrightarrow{JX}$$
$$= -\mathbf{a} + 2(\mathbf{a} + \mathbf{c})$$
$$= \mathbf{a} + 2\mathbf{c}$$

Example 4

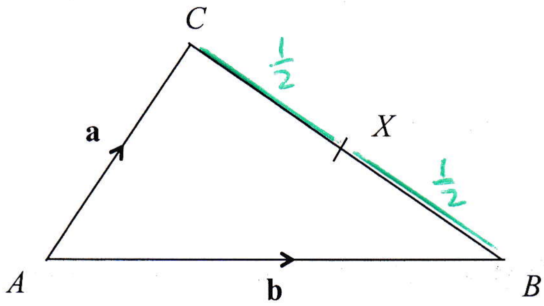


Diagram **NOT** accurately drawn

ABC is a triangle.

$\overrightarrow{AC} = \mathbf{a}$ and $\overrightarrow{AB} = \mathbf{b}$

X is the midpoint of BC IMPORTANT

Find, in terms of **a** and **b**

(i) $\overrightarrow{CB}$

(ii) $\overrightarrow{CX}$

(iii) $\overrightarrow{AX}$

Solution 4:

(i)
$$\overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB}$$
$$= -\mathbf{a} + \mathbf{b}$$

(ii)
$$\overrightarrow{CX} = \frac{1}{2} \times \overrightarrow{CB}$$
$$= \frac{1}{2}(-\mathbf{a} + \mathbf{b})$$
$$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

(iii)
$$\overrightarrow{AX} = \overrightarrow{AC} + \overrightarrow{CX}$$
$$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})$$
$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$

Example 5

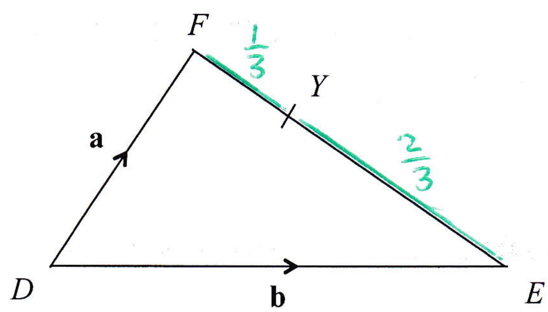


Diagram **NOT** accurately drawn

DEF is a triangle.

$\overrightarrow{DF} = \mathbf{a}$ and $\overrightarrow{DE} = \mathbf{b}$

$\boxed{Y \text{ is the point on } EF \text{ such that } EY : YF = 2:1}$ → USE FRACTIONS $\left[\frac{2}{3} \text{ AND } \frac{1}{3}\right]$

Find, in terms of **a** and **b**

(i) $\overrightarrow{EF}$

(ii) $\overrightarrow{EY}$

(iii) $\overrightarrow{DY}$

Solution 5:

(i) $\overrightarrow{EF} = \overrightarrow{ED} + \overrightarrow{DF}$

$= -b + a$

(ii) $\overrightarrow{EY} = \frac{2}{3}\overrightarrow{EF}$

$= \frac{2}{3}(-b + a)$ → I WOULD WRITE THIS AS $\frac{2}{3}(a - b)$

$= -\frac{2}{3}b + \frac{2}{3}a$

(iii) $\overrightarrow{DY} = \overrightarrow{DE} + \overrightarrow{EY}$

$= b + \frac{2}{3}(a - b)$

$= b + \frac{2}{3}a - \frac{2}{3}b$

$= \frac{2}{3}a + \frac{1}{3}b$

$= \frac{1}{3}(2a + b)$ ← IT IS OFTEN BEST TO FACTORISE THE FINAL EXPRESSION!

Example 6

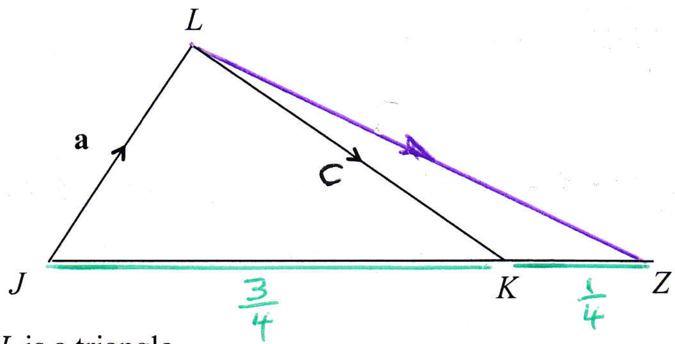


Diagram **NOT** accurately drawn

JKL is a triangle.

$\overrightarrow{JL} = \mathbf{a}$ and $\overrightarrow{LK} = \mathbf{c}$

$\boxed{JK = \frac{3}{4}JZ}$

Find, in terms of **a** and **c**

(i) $\overrightarrow{JK}$

(ii) $\overrightarrow{KZ}$

(iii) $\overrightarrow{LZ}$

Solution 6:

(i) $\overrightarrow{JK} = \overrightarrow{JL} + \overrightarrow{LK}$

$= a + c$

(ii) $\overrightarrow{KZ} = \frac{1}{3} JK$

$= \frac{1}{3}(a+c)$

CAN YOU SEE WHY ITS $\frac{1}{3}$?

(iii) $\boxed{\overrightarrow{LZ}} = \overrightarrow{LK} + \overrightarrow{KZ}$

$= c + \frac{1}{3}(a+c)$

$= c + \frac{1}{3}a + \frac{1}{3}c$

$= \frac{1}{3}a + \frac{4}{3}c$

$= \frac{1}{3}(a+4c)$

Example 7

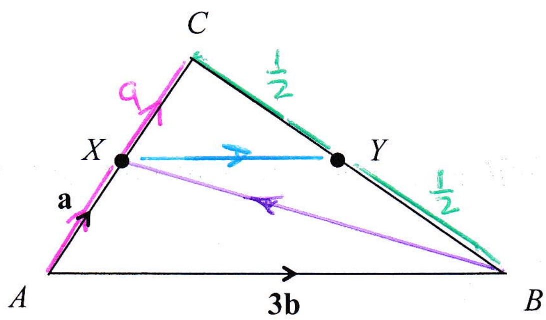


Diagram **NOT** accurately drawn

ABC is a triangle.

X is the midpoint of *AC*

Y is the midpoint of *BC*

$\overrightarrow{AX} = \mathbf{a}$ and $\overrightarrow{AB} = 3\mathbf{b}$

(a) Find, in terms of **a** and **b**

(i) $\overrightarrow{AC}$ ⟶ $\overrightarrow{AC} = 2a$

(ii) $\overrightarrow{BC}$

(iii) $\overrightarrow{BY}$

(iv) $\overrightarrow{BX}$

(v) $\overrightarrow{XY}$

(b) Use a vector method to show that *XY* is parallel to *AB* and that $XY = \frac{1}{2}AB$.

Solution 7:

(ii) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$
$= -3b + 2a$
$= 2a - 3b$

(iii) $\overrightarrow{BY} = \frac{1}{2}\overrightarrow{BC}$
$= a - 1.5b$

(iv) $\boxed{\overrightarrow{BX}} = \overrightarrow{BA} + \overrightarrow{AX}$
$= -3b + a$
$= a - 3b$

(v) $\boxed{\overrightarrow{XY}} = \overrightarrow{XB} + \overrightarrow{BY}$
$= -(a - 3b) + (a - 1.5b)$
$= -a + 3b + a - 1.5b$
$= +1.5b$

(b) SINCE $\overrightarrow{XY} = 1.5b$ AND
$\overrightarrow{AB} = 3b$ THEN...

XY IS PARALLEL TO AB
[BOTH MULIPLES OF b]

AND $XY = \frac{1.5}{3} \times AB$

$= \frac{1}{2}AB$

Example 8

The diagram shows a parallelogram $WXYZ$.

T is the midpoint of XZ

E is the point such that $WE = \frac{1}{3}WX$

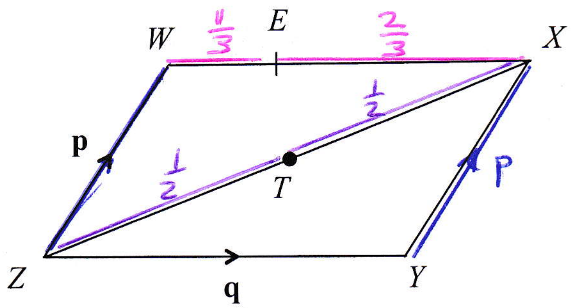


Diagram **NOT** accurately drawn

$\overrightarrow{ZW} = \mathbf{p}$ and $\overrightarrow{ZY} = \mathbf{q}$

Find, in terms of **p** and **q**

(i) $\overrightarrow{ZX}$ $= \overrightarrow{ZY} + \overrightarrow{YX} = q + p$ $\quad (p+q)$

(ii) $\overrightarrow{ZT}$ $\quad \overrightarrow{ZT} = \frac{1}{2}\overrightarrow{ZX} = \frac{1}{2}(p+q)$

(iii) $\overrightarrow{WE}$

(iv) $\overrightarrow{ZE}$

(v) $\overrightarrow{ET}$

(b) Use a vector method to show that T bisects WY.

Solution 8:

(iii) $\overrightarrow{WE} = \frac{1}{3}\overrightarrow{WX} = \frac{1}{3}q$

(iv) $\overrightarrow{ZE} = \overrightarrow{ZW} + \overrightarrow{WE}$

$= p + \frac{1}{3}q$

(v) $\overrightarrow{ET} = \overrightarrow{EW} + \overrightarrow{WZ} + \overrightarrow{ZT}$

$= -\frac{1}{3}q + (-p) + \frac{1}{2}(p+q)$

$= -\frac{1}{2}p + \frac{1}{6}q$

(b) $\overrightarrow{WT} = \overrightarrow{WZ} + \overrightarrow{ZT}$

$= -p + \frac{1}{2}(p+q)$

$= \underline{\underline{\frac{1}{2}(q-p)}}$

$\overrightarrow{WY} = \overrightarrow{WZ} + \overrightarrow{ZY}$

$= -p + q$

$= \underline{\underline{q - p}}$

SINCE

$\overrightarrow{WT} = \frac{1}{2}\overrightarrow{WY}$

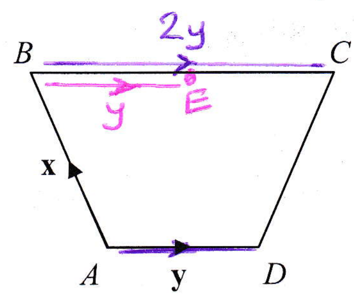
THEN

'T' IS HALFWAY BETWEEN W AND Y

SO 'T' BISECTS WY

PRACTICE QUESTIONS 5

1. The diagram shows a trapezium *ABCD*.



$\overrightarrow{BC} = 2\overrightarrow{AD}$. $= 2y$

$\overrightarrow{AB} = \mathbf{x}$. $\overrightarrow{AD} = \mathbf{y}$.

(a) Find, in terms of **x** and **y**,

(i) $\overrightarrow{AC}$ $= \overrightarrow{AB} + \overrightarrow{BC}$

$= x + 2y$

$x + 2y$ (B1)

(ii) $\overrightarrow{DC}$ $= \overrightarrow{DA} + \overrightarrow{AC}$

$= -y + (x + 2y)$

$= x + y$

$x + y$ (B1)

(2)

(b) The point *E* is such that $\overrightarrow{AE} = \mathbf{x} + \mathbf{y}$.
Use your answer to part (a)(ii) to explain why *AECD* is a parallelogram.

$\overrightarrow{AE} = \overrightarrow{DC}$, SO AE AND DC ARE PARALLEL (B1)

$\overrightarrow{EC} = \frac{1}{2}\overrightarrow{BC}$

$= \frac{1}{2} \times 2y$ (A1) [OTHER METHODS ARE POSSIBLE]

$= y$

$= \overrightarrow{AD}$, SO EC AND AD ARE PARALLEL

(2)

2.

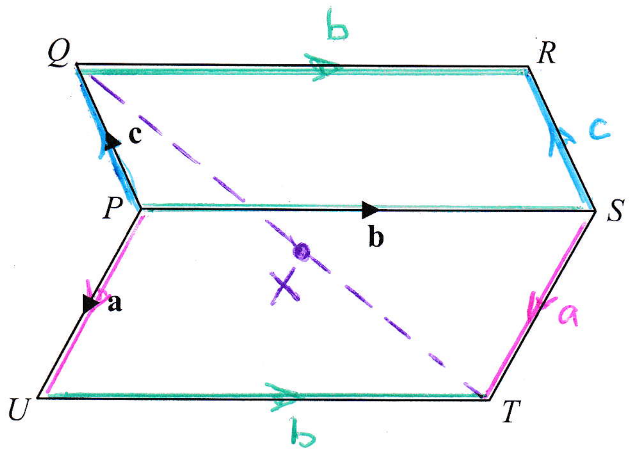


Diagram **NOT** accurately drawn

PQRS and *PSTU* are parallelograms.

$\overrightarrow{PU} = \mathbf{a}$ $\overrightarrow{PS} = \mathbf{b}$ $\overrightarrow{PQ} = \mathbf{c}$

Find, in terms of **a**, **b** and **c**

(i) $\overrightarrow{TQ}$ $= \overrightarrow{TU} + \overrightarrow{UP} + \overrightarrow{PQ}$

$= -b - a + c$

$c - a - b$

B1

(ii) $\overrightarrow{PX}$ where X is the midpoint of TQ.

Simplify your answer as much as possible.

$\overrightarrow{PX} = \overrightarrow{PU} + \overrightarrow{UT} + \frac{1}{2}\overrightarrow{TQ}$

$= a + b + \frac{1}{2}(c - a - b)$ M1

$= a + b + \frac{1}{2}c - \frac{1}{2}a - \frac{1}{2}b$

$= \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$

A1

$\frac{1}{2}(a + b + c)$

3. *PQRSTU* is a regular hexagon, centre *O*.
The hexagon is made from six equilateral triangles of side 2.5 cm.

$\overrightarrow{TU} = \mathbf{a}$. $\overrightarrow{UP} = \mathbf{b}$.

(a) Find, in terms of **a** and/or **b**, the vectors

(i) $\overrightarrow{TP}$ $= \overrightarrow{TU} + \overrightarrow{UP}$

$= a + b$

$a + b$ A1

(1)

(ii) $\overrightarrow{PO}$

$-a$ A1

(1)

(iii) $\overrightarrow{UO}$ $= \overrightarrow{UP} + \overrightarrow{PO}$

$= b - a$

$b - a$ A1

(1)

(b) Find the modulus (magnitude) of $\overrightarrow{UR}$.

2×2.5

5 cm A1

(1)

4. *OABC* is a parallelogram.

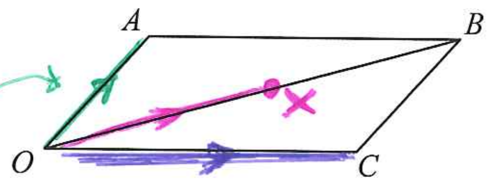


Diagram **NOT** accurately drawn

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

(a) Find the vector $\overrightarrow{OB}$ as a column vector.

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ A1

(1)

X is the point on *OB* such that $OX = kOB$, where $0 < k < 1$

(b) Find, in terms of *k*, the vectors

(i) $\overrightarrow{OX}$, $= k \times \overrightarrow{OB} = k\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5k \\ 2k \end{pmatrix}$ A1

(ii) $\overrightarrow{AX}$, $= \overrightarrow{AO} + \overrightarrow{OX} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 5k \\ 2k \end{pmatrix} = \begin{pmatrix} 5k-1 \\ 2k-2 \end{pmatrix}$ A1

(iii) $\overrightarrow{XC}$. $= \overrightarrow{XO} + \overrightarrow{OC}$

$= -\begin{pmatrix} 5k \\ 2k \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4-5k \\ -2k \end{pmatrix}$ A1

(3)

(c) Find the value of *k* for which $\overrightarrow{AX} = \overrightarrow{XC}$.

$\begin{pmatrix} 5k-1 \\ 2k-2 \end{pmatrix} = \begin{pmatrix} 4-5k \\ -2k \end{pmatrix} \Rightarrow 5k-1 = 4-5k$ M1

$\Rightarrow 10k = 5 \longrightarrow k = \frac{1}{2}$ A1

(2)

(d) Use your answer to part (c) to show that the diagonals of the parallelogram *OABC* bisect one another.

SINCE $k = \frac{1}{2}$, X IS MIDPOINT OF $\overrightarrow{OB}$ M1

$\overrightarrow{AX} = \begin{pmatrix} 5 \times 0.5 - 1 \\ 2 \times 0.5 - 2 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix}$

$\overrightarrow{XC} = \begin{pmatrix} 4 - 5 \times 0.5 \\ -2 \times 0.5 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix}$

SAME SO X IS MIDPOINT OF AC M1

5. PQR is a triangle.
E is the point on PR such that $PR = 3PE$.
F is the point on QR such that $QR = 3QF$.

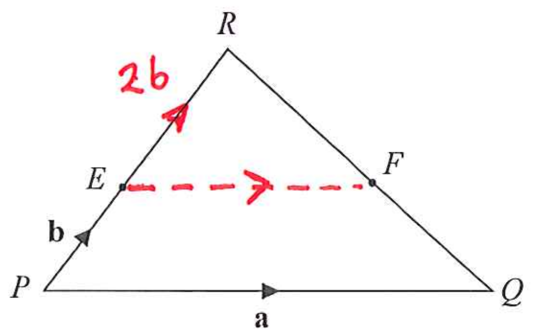


Diagram **NOT** accurately drawn

$\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PE} = \mathbf{b}$.

(a) Find, in terms of **a** and **b**,

(i) $\overrightarrow{PR}$

$3b$ A1

(ii) $\overrightarrow{QR}$

$\overrightarrow{QP} + \overrightarrow{PR} = -a + 3b$

$-a + 3b$ A1

(iii) $\overrightarrow{PF}$

$\overrightarrow{PQ} + \frac{1}{3}\overrightarrow{QR} = a + \frac{1}{3}(-a + 3b)$

$= \frac{2}{3}a + b$

$\frac{2}{3}a + b$ A1

(3)

(b) Show that $\overrightarrow{EF} = k\,\overrightarrow{PQ}$ where k is an integer.

ERROR
k IS NOT AN INTEGER!!

$\overrightarrow{EF} = \overrightarrow{EP} + \overrightarrow{PF}$ M1

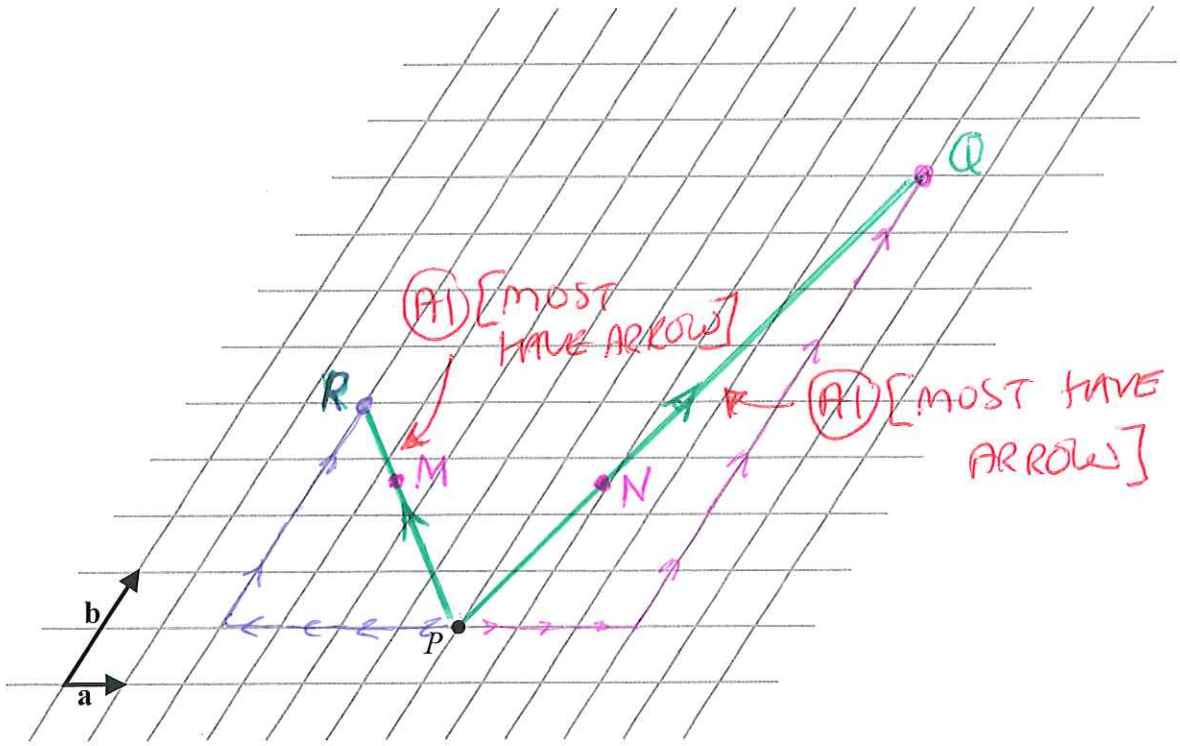
$= -b + \frac{2}{3}a + b$

$= \frac{2}{3}a$ A1

$= k\,\overrightarrow{PQ}$ WHERE $k = \frac{2}{3}$

(2)

6. The diagram shows a grid of equally spaced parallel lines.
The point P and the vectors **a** and **b** are shown on the grid.



$\overrightarrow{PQ} = 3\mathbf{a} + 4\mathbf{b}$

(a) On the grid, mark the vector $\overrightarrow{PQ}$

(1)

$\overrightarrow{PR} = -4\mathbf{a} + 2\mathbf{b}$

(b) On the grid, mark the vector $\overrightarrow{PR}$

(1)

(c) Find, in terms of **a** and **b**, the vector $\overrightarrow{QR}$

$\overrightarrow{QR} = -4b - 3a - 4a + 2b$

$= -7a - 2b$

$\overrightarrow{QR} =$ $-7a - 2b$ A1

(1)

7. PQR is a triangle.
M and N are the midpoints of PQ and PR respectively.

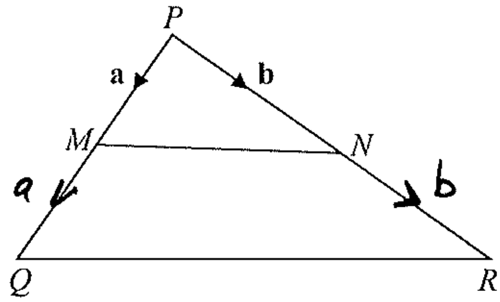


Diagram **NOT** accurately drawn

$\overrightarrow{PM} = \mathbf{a}$ $\overrightarrow{PN} = \mathbf{b}$.

(a) Find, in terms of **a** and/or **b**,

(i) $\overrightarrow{MN} = \overrightarrow{MP} + \overrightarrow{PN}$
$= -a + b$

$b - a$

(ii) $\overrightarrow{PQ} = \overrightarrow{PM} + \overrightarrow{MQ}$
$= 2a$

$2a$

(iii) $\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$
$= -2a + 2b$
$= 2b - 2a$
$= 2(b-a)$

$2(b-a)$

(3)

(b) Use your answers to (a)(i) and (iii) to write down two geometrical facts about the lines MN and QR.

$\overrightarrow{QR} = 2\overrightarrow{MN}$ SO QR IS PARALLEL TO MN AND

QR IS TWICE THE LENGTH OF MN

(2)

8.

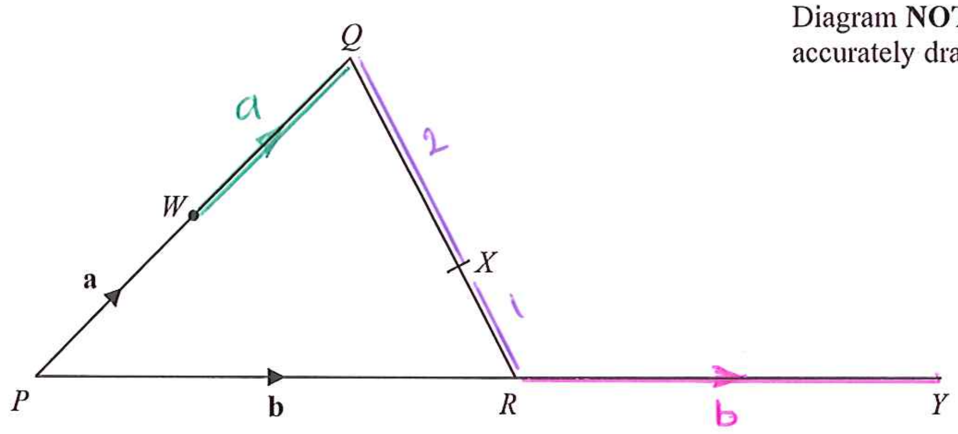


Diagram **NOT** accurately drawn

PQR is a triangle.
The midpoint of *PQ* is *W*.
X is the point on *QR* such that *QX* : *XR* = 2 : 1
PRY is a straight line.

$\overrightarrow{PW} = \mathbf{a}$ $\overrightarrow{PR} = \mathbf{b}$

(a) Find, in terms of **a** and **b**,

(i) $\overrightarrow{QR}$

$= \overrightarrow{QP} + \overrightarrow{PR} \quad = -2a + b$

$b - 2a$ (A1)

(ii) $\overrightarrow{QX}$

$= \frac{2}{3}\overrightarrow{QR} \quad = \frac{2}{3}(b - 2a)$

$\frac{2}{3}b - \frac{4}{3}a$ (A1)

(iii) $\overrightarrow{WX}$

$= \overrightarrow{WQ} + \overrightarrow{QX} \quad = a + \left[\frac{2}{3}b - \frac{4}{3}a\right]$

$\frac{2}{3}b - \frac{1}{3}a$ (A1)

(3)

R is the midpoint of the straight line *PRY*.

(b) Use a vector method to show that *WXY* is a straight line.

$\overrightarrow{XY} = \overrightarrow{XR} + \overrightarrow{RY}$

$= \frac{1}{3}\overrightarrow{QR} + \overrightarrow{RY}$

$= \frac{1}{3}\times[b - 2a] + b$

$= \frac{1}{3}b - \frac{2}{3}a + b$ (B1)

$= \frac{4}{3}b - \frac{2}{3}a \quad = \frac{2}{3}(2b - a)$

NOTE THAT $\overrightarrow{WX} = \frac{1}{3}(2b - a)$ (B1)

BOTH ARE MULTIPLES OF $(2b - a)$ ∴ SAME DIRECTION

ALSO, THEY BOTH GO THROUGH COMMON POINT X

(∴ STRAIGHT LINE)